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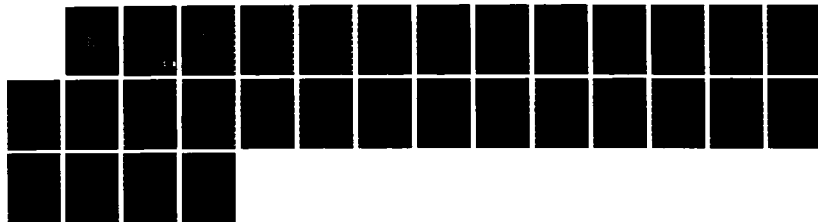
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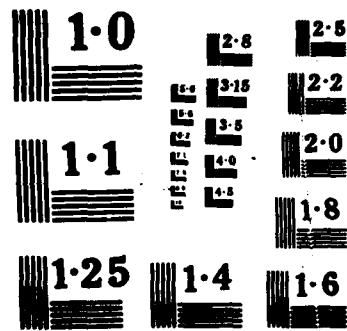
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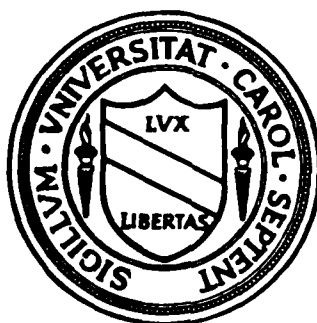
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ON THE PRICING OF AMERICAN OPTIONS

by

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MATTHEW J. KERPER

Chief, Technical Information Division

# ABSTRACT

The problem of valuation for contingent claims that can be exercised at any time before maturity, such as American options, is discussed in the manner of Bensoussan [1]. We offer an approach which both simplifies and extends the results of the existing theory on this topic.



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## 1. Introduction

In an important and relatively recent article [1], A. Bensoussan presents a rigorous treatment of the pricing problem for contingent claims that can be exercised at any time before maturity. He adapts to this situation the Black & Scholes [3] methodology of duplicating the cash flow from such a claim by managing skillfully a self-financing portfolio that contains only the basic instruments of the market, i.e., the stocks and the bond, and that entails no arbitrage opportunities before exercise. Under a condition on the market model called completeness (due to Harrison & Pliska [7],[8] in its full generality and rendered more transparent in [1]), Bensoussan shows that the valuation of such claims is indeed possible and characterizes the exercise time in terms of an appropriate optimal stopping problem.

In the study of the latter, Bensoussan employs the so-called "penalization method," which forces rather stringent boundedness and regularity conditions on the payoff from the contingent claim. Such conditions are not satisfied, however, by the prototypical examples of such claims, i.e., American call options.

The aim of the ~~present~~ paper is to offer an alternative methodology on this problem, which is actually simpler and manages to remove the above restrictions. Furthermore, it seems to be well-suited to the handling of claims that are perpetual, i.e., exercisable at any time before the end of the age.

We present a suitably modified version of the Bensoussan model in sections 2 and 3. The analysis is carried out in section 4, culminating with the valuation formulae (4.8), (4.9). Some elementary consequences of those formulae are discussed. We take up the "perpetual" case in section 5.

## 2. The market model

Let us consider a market in which  $n+1$  assets (or "securities") are traded continuously. One of them, called the bond, has a price  $X^{(0)}$  which evolves according to the equation

$$(2.1) \quad dX_t^{(0)} = r_t X_t^{(0)} dt; \quad X_0^{(0)} = 1$$

with interest rate process  $\{r_t; 0 \leq t < \infty\}$ , and determines the discount factor

$$(2.2) \quad \beta_t \triangleq \frac{1}{X_t^{(0)}} = \exp\left\{-\int_0^t r_s ds\right\}; \quad 0 \leq t < \infty.$$

The remaining  $n$  assets, called the stocks, are risky; their prices are modelled by the linear stochastic differential equations

$$(2.3) \quad \begin{aligned} dX_t^{(i)} &= [a_i(t) - \mu(t)] X_t^{(i)} dt + X_t^{(i)} \sum_{j=1}^d \sigma_{ij}(t) dW_t^{(j)}, \quad 0 \leq t < \infty, \quad 1 \leq i \leq n \\ X_0^{(i)} &= x_i > 0 \end{aligned}$$

with random appreciation rates  $\{a_i(t); 0 \leq t < \infty\}$  and dividend rates  $\{\mu(t); 0 \leq t < \infty\}$  (payable to stockholders). The discounted prices  $\beta X^{(i)}$  of the stocks obey the equations

$$(2.4) \quad d(\beta_t X_t^{(i)}) = \beta_t X_t^{(i)} \left[ \{a_i(t) - r_t - \mu(t)\} dt + \sum_{j=1}^d \sigma_{ij}(t) dW_t^{(j)} \right]; \quad 0 \leq t < \infty, \quad 1 \leq i \leq n.$$

Here, the process  $W = \{W_t = (W_t^{(1)}, \dots, W_t^{(d)})^*, F_t^W; 0 \leq t < \infty\}$  is a standard,  $d$ -dimensional Brownian motion on the space  $(\Omega, F_\infty^W, P)$ . We shall denote by  $\{F_t\}$  the augmentation under  $P$  of the filtration

$$(2.5) \quad F_t^W = \sigma(W_s; 0 \leq s \leq t), \quad 0 \leq t < \infty$$

generated by the Brownian motion; it is well-known (e.g. [10], section 2.7) that  $\{F_t\}$  satisfies the usual conditions of right-continuity and completeness by the  $P$ -null events in  $F_\infty^W$ . One can think of the integer  $d$  as representing



the number of independent, exogeneous sources of uncertainty in the market model. We shall assume that

$$(2.6) \quad \begin{cases} \text{the processes } \{r_t; 0 \leq t < \infty\}, \{a_i(t); 0 \leq t < \infty\}, \{\sigma_{ij}(t); 0 \leq t < \infty\} \\ \text{and } \{\mu(t); 0 \leq t < \infty\} \text{ are progressively measurable with respect to} \\ \{F_t^W\} \text{ and uniformly bounded in absolute value by a constant } C > 0. \end{cases}$$

Let us consider now the random matrices

$$\underline{\sigma}(t) = \{\sigma_{ij}(t)\}_{1 \leq i \leq n, 1 \leq j \leq d}, \quad \underline{D}(t) = \underline{\sigma}(t)\underline{\sigma}^*(t), \quad \underline{H}(t) = \underline{\sigma}^*(t)\underline{\sigma}(t)$$

and the random vector  $\underline{a}(t) - r_t \underline{1}$  with components  $a_i(t) - r_t; 1 \leq i \leq n, 0 \leq t < \infty$ .

2.1 Definition: We shall say that the market model (2.1), (2.3) is complete if there exist positive numbers  $\epsilon, \delta$  such that

$$(2.7) \quad \sum_{i=1}^n \sum_{k=1}^d D_{ik}(t, \omega) \xi_i \xi_k \geq \epsilon \|\underline{\xi}\|^2, \quad \text{for all } \underline{\xi} \in \mathbb{R}^n$$

$$(2.8) \quad \sum_{j=1}^d \sum_{\ell=1}^d H_{j\ell}(t, \omega) \zeta_j \zeta_\ell \geq \delta \|\underline{\zeta}\|^2, \quad \text{for all } \underline{\zeta} \in \mathbb{R}^d$$

hold for every  $(t, \omega) \in [0, \infty) \times \Omega$ . □

Condition (2.7) will be needed below in the construction of the auxiliary probability measure  $\tilde{P}$ , and condition (2.8) in the construction of a "hedging portfolio" (section 4). In particular, (2.7) and (2.8) imply  $n=d$ , i.e., that there exist exactly as many stocks as independent sources of uncertainty in the model. Under completeness, the matrix  $\underline{D}(t, \omega)$  is invertible for every pair  $(t, \omega)$  and the  $\mathbb{R}^d$ -valued process

$$(2.9) \quad \underline{\theta}(t) \triangleq \underline{\sigma}^*(t) \underline{D}^{-1}(t) [\underline{a}(t) - r_t \underline{1}], \quad F_t^W; \quad 0 \leq t < \infty$$

has components which are progressively measurable, uniformly bounded by a constant  $C > 0$ , and satisfy

$$(2.10) \quad \sum_{j=1}^d \sigma_{ij}(t) \theta_j(t) = a_i(t) - r_t; \quad 0 \leq t < \infty, \quad 1 \leq i \leq n$$

everywhere. But then

$$(2.11) \quad Z_t = \exp\left\{-\sum_{j=1}^d \int_0^t \theta_j(s) dW_s^{(j)} - \frac{1}{2} \int_0^t \|\theta(s)\|^2 ds\right\}, F_t; 0 \leq t < \infty$$

is a martingale under  $P$ , and the Girsanov theorem guarantees the existence of a new probability measure  $\tilde{P}$  on  $(\Omega, F_\infty^W)$  which satisfies

$$(2.12) \quad \tilde{P}(F) = E[Z_T 1_F]; \text{ for all } F \in F_T^W$$

for every  $0 \leq T < \infty$ ; under this measure, the process

$$(2.13) \quad \tilde{B}_t \triangleq \tilde{W}_t + \int_0^t \theta(s) ds, F_t^W; 0 \leq t < \infty$$

is a standard Brownian motion in  $\mathbb{R}^d$  (see [9], pp. 176-180 or [10], section 3.5).

Furthermore, for every fixed, finite  $T > 0$ , the probability measure

$$(2.14) \quad \tilde{P}_T(F) \triangleq E[Z_T 1_F]; F \in F_T$$

agrees with  $\tilde{P}$  on  $F_T^W$ , is mutually absolutely continuous with respect to  $P$  on  $F_T$ , and

$$(2.15) \quad \{\tilde{B}_t, F_t; 0 \leq t \leq T\} \text{ is Brownian motion under } \tilde{P}_T.$$

Under this measure, (2.3) and (2.4) can be written, respectively, as

$$(2.16) \quad dX_t^{(i)} = (r_t - \mu(t)) X_t^{(i)} dt + X_t^{(i)} \sum_{j=1}^d \sigma_{ij}(t) dB_t^{(j)}; 0 \leq t \leq T$$

$$(2.17) \quad d(\beta_t X_t^{(i)}) = \beta_t X_t^{(i)} \left[ \sum_{j=1}^d \sigma_{ij}(t) dB_t^{(j)} - \mu(t) dt \right]; 0 \leq t \leq T.$$

**2.2 Remark:** It is seen from (2.17) that the discounted price process

$\{\beta_t X_t^{(i)}, F_t; 0 \leq t \leq T\}$ , for a stock which pays no dividends, is a martingale under the measure  $\tilde{P}_T$ . In fact,  $\tilde{P}$  was constructed with an eye towards this property; see Harrison & Pliska [7],[8] for an amplification of this point. The existence of a probability measure under which the discounted prices become martingales plays a central rôle in the theory of continuous trading

developed by these authors.

More generally, if the dividend rate process  $\mu$  is nonnegative, then the discounted price process  $\beta X^{(i)}$  is a supermartingale under  $\tilde{P}_T$ .  $\square$

Let us denote now by  $\{M_t\}$  the augmentation of the filtration  $\{F_t^W\}$  under  $\tilde{P}$ , and define

$$(2.18) \quad \tilde{F}_t \triangleq M_{t+} = \bigcap_{\epsilon > 0} M_{t+\epsilon}; \quad 0 \leq t < \infty.$$

This new filtration satisfies obviously the usual conditions for  $\tilde{P}$ , and it is not hard to see that

$$(2.19) \quad \{B_t, \tilde{F}_t; 0 \leq t < \infty\} \text{ is a standard Brownian motion under } \tilde{P}.$$

Indeed, we only have to verify that, for every  $f \in C_0^2(\mathbb{R}^d)$ , the process

$$M_t^f \triangleq f(B_t) - f(B_0) - \frac{1}{2} \int_0^t \Delta f(B_s) ds; \quad 0 \leq t < \infty$$

is an  $\{\tilde{F}_t\}$ -martingale under  $\tilde{P}$ . We know from (2.13) that it is an  $\{F_t^W\}$ -martingale; thus, with  $0 < s < s + \frac{1}{n} < t < \infty$ , there exists for every given

$F \in M_{s+\frac{1}{n}}$  an event  $G \in F_{s+\frac{1}{n}}^W$  such that  $\tilde{P}(G \Delta F) = 0$ , and

$$(2.20) \quad \tilde{E}[(M_t^f - M_{s+\frac{1}{n}}^f)1_F] = \tilde{E}[(M_t^f - M_{s+\frac{1}{n}}^f)1_G] = 0.$$

By taking  $F \in M_{s+} = \tilde{F}_s$  and then letting  $n \rightarrow \infty$  in (2.20), we obtain

$$\tilde{E}[(M_t^f - M_s^f)1_F] = 0, \text{ and therefore (2.19) as well.}$$

**2.3 Remark:** All the processes under consideration are adapted to  $\{\tilde{F}_t\}$ , and the equations (2.16), (2.17) are valid for  $0 \leq t < \infty$  on  $(\Omega, \tilde{F}_\infty, \tilde{P})$ .

### 3. Contingent claims and equivalent portfolios

In order to fix ideas, let us take  $d=n=1$  in the market model of the previous section, and suppose that at time  $t=0$  we sign a contract which gives us the option to buy, at any time  $\tau$  between  $t=0$  and an "expiration date"  $t=T$ , one share of the stock at a specified price of  $c$  dollars (the contractual "exercise price"). If the price  $X_\tau^{(1)}$  of the stock is below the exercise price at  $t=\tau$ , the contract is worthless to us; but if  $X_\tau^{(1)} > c$ , we can exercise our option (i.e., to buy one share at the preassigned price  $c$ ) and then sell the share immediately in the market, thus making a net profit of  $(X_\tau^{(1)} - c)^+$  dollars. Because clairvoyance has to be excluded,  $\tau$  is restricted to be a stopping time of  $\{F_t\}$  with values in  $[0, T]$ .

Such a contract is commonly called an American option, in contradistinction to "European options" which allow exercise only on the expiration date, i.e.,  $\tau=T$ . Both European and American options are financial instruments and can be traded on their own right (e.g. at the Chicago Board Options Exchange and other organized secondary markets for options).

Two related questions can be raised for such instruments:

- (3.1) (i) When should an American option be exercised, if at all?
- (3.1) (ii) How much should one be willing to pay at  $t=0$  for the right to sign the abovementioned contract?

We shall see that the key to answering both these questions comes in the form of an appropriate optimal stopping problem. The following definition generalizes the concept of American option; we denote by  $S_{u,v}$  the collection of all stopping times  $\tau$  of  $\{F_t\}$  with values in  $[u, v]$ , for fixed  $0 \leq u < v \leq \infty$ , and write  $S_t^* \triangleq S_{t, \infty}$ ,  $S_t \triangleq \bigcup_{n=1}^{\infty} S_{t, t+n}$ .

**3.1 Definition:** An American contingent claim (ACC) is a financial instrument specified by

- (i) an expiration date  $T \in (0, \infty]$ ,
- (ii) the selection of an exercise time  $\tau \in S_{0,T}$ ,
- (iii) a payoff rate  $g_t$  per unit time on  $(0, \tau)$ , and
- (iv) a terminal payoff  $f_\tau$  at the exercise time.

The processes  $F = \{f_t, F_t^W; 0 \leq t < \infty\}$  and  $G = \{g_t, F_t^W; 0 \leq t < \infty\}$  are non-negative, progressively measurable, and satisfy for some  $m > 1$ :

$$(3.2) \quad E \left[ \sup_{0 \leq t \leq T} f_t + \int_0^T g_t dt \right]^m < \infty, \text{ for every fixed } T \in (0, \infty).$$

Furthermore,  $F$  is assumed to have continuous paths. □

**3.2 Remark:** An ACC with expiration date  $T = \infty$  is referred to as perpetual; in this case, (iv) above is to be understood with the convention

$$f_\infty(\omega) \triangleq \overline{\lim}_{t \rightarrow \infty} f_t(\omega), \quad \omega \in \Omega.$$

**3.3 Example:** An American option is a special case of an ACC with  $n = d = 1$ ,  $g_t \equiv 0$  and  $f_t = (X_t^{(1)} - c)^+$ . The number  $c \geq 0$  is called the exercise price of the option.

Let us suppose now that at time  $t = 0$  we sign a contract which entitles us to an ACC. What is then the value  $V_t$  of this instrument to us, for every  $0 \leq t \leq T$ ? If we can answer this question, then we can also answer question 3.1(ii) above: we should be willing to accept at  $t = 0$  a fee which is proportional to  $V_0$ . This is the essence of the contingent claim valuation problem, which has a long history (see Samuelson [14], [15], McKean [11], Van Moerbeke [17] and the review article by Smith [16]). We shall approach here

in the spirit of Merton [12], Harrison & Pliska [7],[8] and particularly Bensoussan [1]. The fundamental idea, due to Black & Scholes [3] in the setting of European options, is to try to duplicate the cash flow from an ACC by skillfully managing a portfolio that contains shares from the basic instruments of the market, i.e., the stocks and the bond.

3.4 Definition: A vector  $\Pi = \{(\pi_0(t), \pi_1(t), \dots, \pi_n(t)), F_t; 0 \leq t < \infty\}$  of measurable and adapted processes on  $(\Omega, \mathcal{F})$  with

$$(3.3) \quad P\left[\int_0^T \pi_i^2(s) ds < \infty\right] = 1; \quad 0 \leq T < \infty, \quad i = 0, 1, \dots, n$$

is called a portfolio; its components represent the number of shares, from each of the  $n+1$  assets, that are to be held (if  $\pi_i(t) \geq 0$ ) or borrowed (if  $\pi_i(t) < 0$ ) at time  $t$ . The quantity

$$(3.4) \quad V_t = \sum_{i=0}^n \pi_i(t) X_t^{(i)}; \quad 0 \leq t < \infty$$

is called the value of the portfolio at that time. □

The question now is to choose the portfolio  $\Pi$  in such a way as to imitate the cash flow from the ACC; then  $V_t$  in (3.4) will represent not only the value of the portfolio but also that of the contingent claim for  $0 \leq t \leq T$ . For this to happen, we should have necessarily

$$(3.5) \quad V_t \geq f_t; \quad \text{for all } 0 \leq t < T$$

and, if  $T < \infty$ , also

$$(3.6) \quad V_T = f_T$$

almost surely, because we always have the option of exercising our claim (and only this option if  $t = T < \infty$ ). On the other hand, we want to build the portfolio in such a way that the total earnings (i.e., capital gains plus dividend payments)

$$\sum_{i=0}^n \int_t^{t+h} \pi_i(s) dX_s^{(i)} + \sum_{i=1}^n \int_t^{t+h} \pi_i(s) X_s^{(i)} \mu_i(s) ds$$

from it over any interval  $(t, t+h)$  should not fall below the potential earnings

$$V_{t+h} - V_t + \int_t^{t+h} g_s ds$$

that would result from the possession of the claim. This is the so-called hedging property of the portfolio; we impose it by postulating that the process

$$(3.7) \quad \left\{ \begin{aligned} A_t &\triangleq \sum_{i=0}^n \int_0^t \pi_i(s) dX_s^{(i)} + \sum_{i=1}^n \int_0^t \pi_i(s) X_s^{(i)} \mu_i(s) ds - V_t - \int_0^t g_s ds + V_0 \\ &\text{is almost surely nondecreasing.} \end{aligned} \right.$$

Finally, let us introduce for every  $t \in [0, T)$  the random time

$$(3.8) \quad \hat{\tau}_t = \inf\{s \geq t; V_s = f_s\}$$

and observe from (3.5), (3.6) that  $t \leq \hat{\tau}_t \leq T$  holds almost surely, if  $T < \infty$ . It is certainly plausible that, if we can guarantee the stopping time property of  $\hat{\tau}_t$ , this time will then be the best amongst all  $\tau \in S_{t, T}$  to exercise the claim. Therefore, on the interval  $(t, \hat{\tau}_t)$  the situation is very much like that for a European claim: the gains from the portfolio and the gains from the claim should coincide, so that no arbitrage opportunities could exist. Equivalently, the process  $A$  of (3.7) should be constant on this interval:

$$(3.9) \quad A_t = A_{\hat{\tau}_t}, \text{ almost surely.}$$

We shall see in the next section that it is possible to construct a portfolio with all these properties. The following technical notion and result will be needed.

3.5 Definition: A measurable, adapted process  $\{Y_t, F_t; 0 \leq t < \infty\}$  is said to be of

- (1) class D, if the family  $\{Y_\tau; \tau \in S_0\}$  is uniformly integrable;

- (ii) class  $D[0, T]$ , for a given  $0 < T < \infty$ , if the family  $\{Y_\tau; \tau \in S_{0, T}\}$  is uniformly integrable;
- (iii) class  $DL$ , if it is of class  $D[0, T]$  for every  $0 < T < \infty$ .

3.6 Lemma: The progressively measurable process

$$(3.10) \quad Q_t \triangleq f_t \beta_t + \int_0^t g_s \beta_s ds, \quad F_t^W; \quad 0 \leq t < \infty$$

is of class  $DL$  under  $\tilde{P}$ .

Proof: With  $p \triangleq \sqrt{m} > 1$ ,  $K_T \triangleq e^{CT} (\max_{0 \leq t \leq T} f_t + \int_0^T g_s ds)$  we have from (2.12) and the Hölder inequality:

$$(3.11) \quad \tilde{E}(\max_{0 \leq t \leq T} Q_t)^p \leq \tilde{E}(K_T^p) = E(K_T^p Z_T) \leq (EK_T^m)^{1/p} (EZ_T^q)^{1/q},$$

where  $1/p + 1/q = 1$ . Now (3.2) gives  $EK_T^m < \infty$ , and for any  $q > 1$  we can write

$$Z_T^q = \exp\left\{-\sum_{j=1}^d \int_0^T q \hat{\psi}_j(s) dW_s^{(j)} - \frac{1}{2} \int_0^T \|q \hat{\psi}(s)\|^2 ds\right\} \cdot \exp\left\{\frac{q(q-1)}{2} \int_0^T \|\hat{\psi}(s)\|^2 ds\right\},$$

whence

$$EZ_T^q \leq \exp\left\{\frac{1}{2} q(q-1) TC^2\right\} < \infty.$$

From (3.11) we obtain  $\sup_{\tau \in S_{0, T}} \tilde{E}Q_\tau^p < \infty$ , and the requisite uniform integrability follows.



#### 4. Claims with finite expiration date

We broach now the question of valuating an ACC with  $T < \infty$ ; all the processes under consideration will be defined only on  $[0, T]$ .

**4.1 Definition:** A portfolio  $\Pi$  is called a hedging portfolio against the ACC if its value  $V$  is continuous, of class  $D[0, T]$ , and satisfies (3.5), (3.6), (3.7) and (3.9), under  $\tilde{P}_T$ .

If such a portfolio exists, its value  $V$  is called a valuation process for the ACC, and  $V_0$  is referred to as a value of the ACC at  $t = 0$ .  $\square$

Because of the continuity of both  $V$  and  $F$ , the random time  $\hat{t}_t$  of (3.8) is indeed a stopping time of  $\{F_t\}$ . We shall show in Theorem 4.2 below that a valuation process exists and is unique up to indistinguishability.

Let us start by assuming the existence of such a process ; recalling (2.1), (2.16), (3.4) we can rewrite the process  $A$  of (3.7) in the equivalent form

$$(4.1) \quad A_t = V_0 - V_t + \int_0^t (r_s V_s - g_s) ds + \sum_{i=1}^n \sum_{j=1}^d \int_0^t \pi_i(s) X_s^{(i)} \sigma_{ij}(s) dB_s^{(j)}; \quad 0 \leq t \leq T.$$

We fix  $t \in [0, T)$ , select a stopping time  $\tau \in S_{t, T}$ , and define

$$(4.2) \quad \tau_m = \tau \wedge \inf\{s \in [t, T]; X_s^{(i)} \geq m \text{ or } \int_t^s \pi_i^2(u) du \geq m, \text{ for some } 1 \leq i \leq n\}$$

for  $m = 1, 2, \dots$ . The equations (2.2), (4.1) give

$$(4.3) \quad \beta_t(A_t + V_t) - \beta_{\tau_m}(A_{\tau_m} + V_{\tau_m}) = \int_t^{\tau_m} \beta_s(g_s + r_s A_s) ds - \sum_{i=1}^n \sum_{j=1}^d \int_t^{\tau_m} \beta_s \pi_i(s) X_s^{(i)} \sigma_{ij}(s) dB_s^{(j)}$$

a.s.  $\tilde{P}_T$ . But the non-decreasing nature of  $A$  implies in particular

$$(4.4) \quad \beta_{\tau_m} A_{\tau_m} - \beta_t A_t + \int_t^{\tau_m} r_s \beta_s A_s ds = \int_t^{\tau_m} \beta_s dA_s \geq 0,$$

which gives in conjunction with (4.3):

$$(4.5) \quad \beta_t V_t \geq \beta_{\tau_m} V_{\tau_m} + \int_t^{\tau_m} \beta_s g_s ds - \sum_{i=1}^n \sum_{j=1}^d \int_t^{\tau_m} \pi_i(s) \beta_s X_s^{(i)} \sigma_{ij}(s) dB_s^{(j)}, \text{ a.s. } \mathbb{P}_T.$$

The conditional expectations (given  $F_t$ ) of the stochastic integrals are equal to zero, and thus

$$(4.6) \quad \beta_t V_t \geq \mathbb{E}_T[\beta_{\tau_m} V_{\tau_m} + \int_t^{\tau_m} \beta_s g_s ds | F_t], \text{ a.s. } \mathbb{P}_T.$$

Now we may let  $m \rightarrow \infty$ ; because of (3.3) we have  $\lim_{m \rightarrow \infty} \tau_m = \tau$ , a.s.  $\mathbb{P}_T$ , and by the monotone convergence theorem, the membership of  $V$  in  $D[0, T]$ , and (3.5), we deduce

$$(4.7) \quad \beta_t V_t \geq \mathbb{E}_T[\beta_{\tau} f_{\tau} + \int_t^{\tau} \beta_s g_s ds | F_t], \text{ a.s. } \mathbb{P}_T$$

for every  $\tau \in S_{t, T}$ . On the other hand, with  $\tau = \hat{\tau}_t$ , all four of (4.4)-(4.7) hold as identities, and thus

$$\beta_t V_t = \mathbb{E}_T[\beta_{\hat{\tau}_t} f_{\hat{\tau}_t} + \int_t^{\hat{\tau}_t} \beta_s g_s ds | F_t], \text{ a.s. } \mathbb{P}_T.$$

**4.2 Theorem:** Under the assumptions (2.6)-(2.8) and (3.2), there exists a continuous, adapted process  $V = \{V_t, F_t; 0 \leq t \leq T\}$  which is a valuation process for the ACC of Definition 3.1 and admits the representation

$$(4.8) \quad V_t = \operatorname{esssup}_{\tau \in S_{t, T}} \mathbb{E}_T[f_{\tau} \exp\{-\int_t^{\tau} r_s ds\} + \int_t^{\tau} g_s \exp\{-\int_t^s r_u du\} ds | F_t], \text{ a.s. } \mathbb{P}_T$$

for every fixed  $t \in [0, T]$ . Every other valuation process is a modification of (and hence indistinguishable from)  $V$ . In particular,

$$(4.9) \quad V_0 = \sup_{\tau \in S_{0, T}} \mathbb{E}_T[f_{\tau} \exp\{-\int_0^{\tau} r_s ds\} + \int_0^{\tau} g_s \exp\{-\int_0^s r_u du\} ds]. \quad \square$$

The uniqueness claim has just been shown; for the existence, let us recall the process  $Q$  of (3.10) and consider the optimal stopping problem of characterizing the function

$$(4.10) \quad u(t) \stackrel{\Delta}{=} \sup_{\tau \in S_{0,T}} \tilde{E}_T(Q_\tau).$$

This problem was treated by Fakeev [5] and Bismut & Skalli [2]; according to the results of these papers, there exists a right-continuous supermartingale  $\xi = \{\xi_t, F_t; 0 \leq t < \infty\}$  such that

$$(4.11) \quad u(t) = \tilde{E}_T(\xi_t)$$

and

$$(4.12) \quad \xi_t = \text{esssup}_{\tau \in S_{t,T}} \tilde{E}_T(Q_\tau | F_t), \quad \text{a.s. } \tilde{P}_T$$

are valid for every  $t \in [0, T]$ . The process  $\xi$  turns out to be the Snell envelope of  $Q$  (i.e., the minimal right-continuous supermartingale which majorizes  $Q$ ), and the stopping time

$$(4.13) \quad \sigma_t \stackrel{\Delta}{=} \inf\{s \geq t; \xi_s = Q_s\} \in S_{t,T}$$

is optimal for every given  $t \in [0, T]$ :

$$(4.14) \quad u(t) = \tilde{E}_T(Q_{\sigma_t}).$$

Moreover, it was shown by Bismut & Skalli [2] that the supermartingale  $\xi$  is regular:

$$(4.15) \quad \begin{cases} \text{for every sequence } \{\sigma_n\}_{n=1}^\infty \subseteq S_{t,T} \text{ converging a.s. } \tilde{P}_T \text{ to a} \\ \text{stopping time } \sigma \in S_{t,T}, \text{ we have } \lim_{n \rightarrow \infty} \tilde{E}_T(\xi_{\sigma_n}) = \tilde{E}_T(\xi_\sigma). \end{cases}$$

4.3 Lemma: For every  $t \in [0, T]$ , the process  $\{\xi_{s \wedge \sigma_t}, F_s; t \leq s \leq T\}$  is a martingale under  $\tilde{P}_T$ .

Proof: The above process is a supermartingale, by the optional sampling theorem; it also has constant expectation, since

$$\tilde{E}_T(\xi_{\sigma_t}) = \tilde{E}_T(Q_{\sigma_t}) = u(t) = \tilde{E}_T(\xi_t)$$

by virtue of (4.11), (4.14). The martingale property follows.

**4.4 Lemma:** The supermartingale  $\xi$  of (4.12) is of class  $D[0, T]$  under  $\tilde{P}_T$ .

**Proof:** We proceed as in the proof of Lemma 3.6; if  $\{m_t; 0 \leq t \leq T\}$  is a right-continuous modification of the martingale  $\{\tilde{E}_T(K_T | F_t); 0 \leq t \leq T\}$  and  $\lambda \stackrel{\Delta}{=} q^P$ , the Doob, Hölder and Jensen inequalities give

$$\tilde{E}_T(\sup_{0 \leq t \leq T} \xi_t^P) \leq \tilde{E}_T(\sup_{0 \leq t \leq T} m_t^P) \leq \lambda \tilde{E}_T(m_T^P) \leq \lambda \tilde{E}_T(K_T^P).$$

The remainder of the proof follows that of Lemma 3.6.  $\square$

Now Lemma 4.4 and regularity (condition (4.15)) show that  $\xi$  admits the Doob-Meyer decomposition (cf. [13], Chapter VII)

$$(4.16) \quad \xi = M - \Lambda$$

where  $\Lambda = \{\Lambda_t, F_t; 0 \leq t \leq T\}$  is a continuous, nondecreasing process with  $\tilde{E}_T(\Lambda_T) < \infty$ , and  $M = \{M_t, F_t; 0 \leq t \leq T\}$  is a right-continuous version of the  $\tilde{P}_T$ -martingale  $\{\tilde{E}_T(M_T | F_t); 0 \leq t \leq T\}$  with  $M_T \stackrel{\Delta}{=} Q_T + \Lambda_T$ . The Bayes rule gives

$$\tilde{E}_T(M_T | F_t) = \frac{E(M_T Z_T | F_t)}{Z_t} = \frac{N_t}{Z_t}, \quad \text{a.s. } P$$

where  $\{N_t, F_t; 0 \leq t \leq T\}$  is a right-continuous version of the  $P$ -martingale  $\{E(M_T Z_T | F_t); 0 \leq t \leq T\}$ . This version can actually be taken as continuous, since by the basic representation theorem ([9], p. 80 or [10], section 3.4) we have

$$(4.17) \quad N_t = E(M_T Z_T) + \sum_{j=1}^d \int_0^t \phi_j(s) dW_s^{(j)}; \quad 0 \leq t \leq T$$

a.s.  $\tilde{P}_T$ , for suitable measurable and adapted processes  $\{\phi_j(t), F_t; 0 \leq t \leq T\}$  such that

$$(4.18) \quad P\left[\int_0^T \phi_j^2(t) dt < \infty\right] = 1; \quad 1 \leq j \leq d.$$

It develops that the continuous process

$$(4.16)' \quad \zeta = \frac{N}{Z} - \Lambda$$

is indistinguishable from  $\xi$ ; furthermore, from Lemma 4.3 and the uniqueness of the Doob-Meyer decomposition, we have for every  $t \in [0, T]$

$$(4.19) \quad \Lambda_t = \Lambda_{\rho_t}, \text{ a.s. } \tilde{\mathbb{P}}_T.$$

It is also pretty obvious from (4.12) and the continuity of both  $\zeta$  and  $Q$ , that

$$(4.20) \quad \zeta_t \geq Q_t, \quad 0 \leq t \leq T$$

$$(4.21) \quad \zeta_T = Q_T$$

hold a.s.  $\tilde{\mathbb{P}}_T$ .

Proof of Theorem 4.2: The process

$$(4.22) \quad V_t \stackrel{\Delta}{=} \frac{1}{\beta_t} (\zeta_t - \int_0^t \beta_s g_s ds), F_t; \quad 0 \leq t \leq T$$

is obviously adapted, continuous and of class  $D[0, T]$  (Lemma 4.4), and satisfies (4.8), (3.5), (3.6) thanks to (4.12), (4.20), (4.21); besides, the stopping times  $\hat{\tau}_{t, \rho_t}$  of (3.8), (4.13) are a.s. equal.

On the other hand, a straightforward application of Itô's rule to (4.16)' yields, in conjunction with (4.17) and (2.11),

$$(4.23) \quad d\zeta_t = \frac{1}{Z_t} \sum_{j=1}^d \{ \sigma_j(t) + N_t \theta_j(t) \} dB_t^{(j)} - d\Lambda_t$$

and then we obtain from (4.22):

$$(4.24) \quad V_t = V_0 + \int_0^t (r_s V_s - g_s) ds + \sum_{j=1}^d \int_0^t \frac{X_s^{(0)}}{Z_s} (\phi_j(s) + N_s \theta_j(s)) dB_s^{(j)} - \int_0^t X_s^{(0)} d\Lambda_s; \quad 0 \leq t \leq T$$

almost surely. But now, comparing (4.1) and (4.24), we conclude that (3.7) will be satisfied with the choice

$$(4.25) \quad A_t \triangleq \int_0^t X_s^{(0)} d\Lambda_s; \quad 0 \leq t \leq T,$$

provided that we select the portfolio  $\Pi$  in such a manner that

$$\beta_t Z_t \sum_{i=1}^n \pi_i(t) X_t^{(i)} \sigma_{ij}(t) = \phi_j(t) + N_t \theta_j(t); \quad 0 \leq t \leq T, \quad 1 \leq j \leq d.$$

This can be accomplished (thanks to (2.8)) by introducing the  $\mathbb{R}^n$ -valued process

$$(4.26) \quad \tilde{\eta}(t) \triangleq \frac{X_t^{(0)}}{Z_t} \sigma(t) H^{-1}(t) [\phi(t) + N_t \theta(t)], F_t; \quad 0 \leq t \leq T$$

and then setting

$$(4.27) \quad \pi_i(t) \triangleq \frac{\eta_i(t)}{X_t^{(i)}}; \quad 1 \leq i \leq n, \quad \pi_0(t) \triangleq \beta_t [V_t - \sum_{i=1}^n \eta_i(t)].$$

It is easily seen from (4.26), (4.18) that this  $\Pi$  satisfies the integrability condition (3.3). The continuous, nondecreasing process  $A$  of (4.25) obeys the condition (3.9) because of (4.19), and the proof is complete.

4.5 Remark: Theorem 4.2 was established in [1] under a regularity condition on the process  $F$  and under the assumption that both processes  $F, G$  of Definition 3.1 are uniformly bounded. This condition is not satisfied, however, in the prototypical case of an American option (Example 3.3).

Let us examine now some elementary consequences of Theorem 4.2.

4.6 Remark: Consider the case where the process  $\{Q_t, F_t; 0 \leq t \leq T\}$  of (3.10) is a submartingale under  $\tilde{P}_T$  (or equivalently, the process  $\{Q_t Z_t, F_t; 0 \leq t \leq T\}$  is a submartingale under  $P$ ). Then it is easily seen from (4.12) and the optional sampling theorem that

$$\xi_t = \tilde{E}_T(Q_T | F_t), \text{ a.s. } \tilde{P}_T \quad \text{and} \quad u(t) = \tilde{E}_T(Q_T)$$

hold for every  $0 \leq t \leq T$ , i.e.,  $\tau_t = T$  is optimal in (4.10). It develops that

the valuation problem is the same, in this case, as that for a European contingent claim:

$$(4.28) \quad V_t = \tilde{E}_T \left[ f_T \exp \left\{ - \int_t^T r_s ds \right\} + \int_t^T g_s \exp \left\{ - \int_t^s r_u du \right\} ds \mid F_t \right], \quad \text{a.s. } \tilde{P}_T.$$

For instance, in the case of Example 3.3 with  $r_t \geq 0$ ,  $\mu(t) \equiv 0$  and  $c > 0$ ,  
the process

$$Q_t = (\beta_t X_t^{(1)} - c\beta_t)^+$$

is easily seen to be a submartingale under  $\tilde{P}_T$ ; cf. Remark 2.2. We recover a result of Merton (1973) in the following form: an American option with positive exercise price, written on a stock which pays no dividends, should not be exercised before the expiration date.

4.7 Remark: If the process  $\{Q_t, F_t; 0 \leq t \leq T\}$  is a supermartingale under  $\tilde{P}_T$ , then  $\xi = Q$ ,  $u(t) = \tilde{E}_T(Q_t)$ ,  $\hat{t}_t = t$  and (4.22), (3.10) give

$$(4.29) \quad V = f.$$

Consider in this vein the situation in Example 3.3 with  $c = 0$ ,  $\mu(t) \geq 0$ . Then  $Q = \beta X^{(1)}$  is a supermartingale under  $\tilde{P}_T$ , and (4.29) gives  $V = X^{(1)}$ . In other words, an American option with zero exercise price must sell for the same amount as the stock.

Finally, in the case of Example 3.3 with  $\mu(t) \geq 0$ , we have from Remark 2.2 and (4.8):

$$\beta_t V_t = \text{esssup}_{\tau \in S_{t,T}} \tilde{E}_T [\beta_\tau (X_\tau^{(1)} - c)^+ \mid F_t] \leq \text{esssup}_{\tau \in S_{t,T}} \tilde{E}_T [\beta_\tau X_\tau^{(1)} \mid F_t] \leq \beta_t X_t^{(1)}$$

a.s.  $\tilde{P}_T$ , i.e.,  $V \leq X^{(1)}$ : the underlying stock is always at least as valuable as the option.

## 5. Perpetual claims

When it comes to studying perpetual ACC's (Remark 3.2) it becomes essential that the Brownian motion  $B$  of (2.13) be defined on the entire of  $[0, \infty)$  and be accompanied by a filtration which satisfies the usual conditions. As such a filtration one could take the  $\tilde{P}$ -augmentation of

$$\mathcal{F}_t^B = \sigma(B_s; 0 \leq s \leq t); 0 \leq t < \infty,$$

but this filtration will typically fail to measure the processes of (2.6).

On the other hand, we may choose the filtration  $\{\tilde{F}_t\}$  of (2.18) for this purpose. Indeed, we shall take in this section  $(\Omega, \tilde{F}_\infty, P), \{\tilde{F}_t\}$

as our basic probability space, and recall (2.19) as well as Remark 2.3.

Similarly, the classes  $S_{u,v}, S_t, S_t^*$  introduced in section 3 will now consist of stopping times of  $\{\tilde{F}_t\}$ . It will also be assumed that

$$(5.1) \quad \tilde{E}(\sup_{0 \leq t < \infty} Q_t) < \infty$$

holds, a condition which guarantees that the process  $Q$  of (3.10) is of class  $D$  under  $\tilde{P}$ .

5.1 Definition: A vector  $\Pi = \{(\pi_0(t), \pi_1(t), \dots, \pi_n(t)), \tilde{F}_t; 0 \leq t < \infty\}$  of measurable and adapted processes on  $(\Omega, \tilde{F}_\infty)$  with

$$(5.2) \quad \tilde{P}\left[\int_0^T \pi_i^2(t) dt < \infty\right] = 1; \quad 0 \leq T < \infty, \quad i = 0, 1, \dots, n$$

will be called a portfolio, and the process  $V = \{V_t, \tilde{F}_t; 0 \leq t < \infty\}$  of (3.4) the value of the portfolio.

5.2 Definition: A portfolio  $\Pi$  is called a hedging portfolio against the perpetual ACC if its value  $V$  is continuous and satisfies (3.5), (3.7), (3.9) as well as

$$(5.3) \quad \lim_{t \rightarrow \infty} \beta_t V_t = \lim_{t \rightarrow \infty} \beta_t f_t, \quad \text{a.s. } \tilde{P},$$



and the discounted value  $BV = \{\beta_t V_t, \tilde{F}_t; 0 \leq t < \infty\}$  is of class D. If such a portfolio exists, its value is called a valuation process for the perpetual ACC, and  $V_0$  is referred to as a value of the perpetual ACC at  $t = 0$ .  $\square$

The condition (5.3) is, in a sense, an analogue of (3.6). For a generic nonnegative process  $Y = \{Y_t, \tilde{F}_t; 0 \leq t < \infty\}$  we shall use the conventions

$$Y_\infty(\omega) \triangleq \overline{\lim_{t \rightarrow \infty}} Y_t(\omega)$$

and  $\tilde{E}Y_T = \tilde{E}[Y_T 1_{\{T < \infty\}} + Y_\infty 1_{\{T = \infty\}}]$ . Proceeding as in section 4 (with  $T = \infty$  in (4.2)) we can establish the following analogue of Theorem 4.2.

**5.3 Theorem:** Under the assumptions (2.6)-(2.8) and (5.1), there exists a valuation process  $V$  for the perpetual ACC, which admits the representation

$$(5.4) \quad V_t = \operatorname{esssup}_{\tau \in S_t^*} \tilde{E}[f_\tau \exp\{-\int_t^\tau r_s ds\} + \int_t^\tau g_s \exp\{-\int_t^s r_u du\} ds | \tilde{F}_t], \text{ a.s. } \tilde{P}$$

for every fixed  $t \in [0, \infty)$ . Every other valuation process is indistinguishable from  $V$ , and

$$(5.5) \quad V_0 = \sup_{\tau \in S_0^*} \tilde{E}[f_\tau \exp\{-\int_0^\tau r_s ds\} + \int_0^\tau g_s \exp\{-\int_0^s r_u du\} ds]. \quad \square$$

The detailed development is omitted; as before, it uses the results of [5],[2] for the optimal stopping problem

$$(5.6) \quad u(t) \triangleq \sup_{\tau \in S_t^*} \tilde{E}(Q_\tau) \equiv \sup_{\tau \in S_t} \tilde{E}(Q_\tau).$$

The Snell envelope for this problem, i.e., the minimal right-continuous supermartingale  $\xi = \{\xi_t, \tilde{F}_t; 0 \leq t < \infty\}$  which dominates  $Q$ , satisfies

$$(5.7) \quad \xi_t = \operatorname{esssup}_{\tau \in S_t^*} \tilde{E}(Q_\tau | \tilde{F}_t) = \operatorname{esssup}_{\tau \in S_t} \tilde{E}(Q_\tau | \tilde{F}_t), \text{ a.s. } \tilde{P}$$

for every  $0 \leq t < \infty$ , and is now regular (i.e., (4.15) holds on every finite

horizon  $[0, T]$ ) and of class D under  $\tilde{P}$ . For the last claim, it is useful to recall the strengthening of (5.7):

$$(5.8) \quad \xi_\sigma = \operatorname{esssup}_{\tau \in S_\sigma} \tilde{E}(Q_\tau | \tilde{F}_\sigma), \quad \text{a.s. } \tilde{P}$$

(e.g. [4], §2.15), which is valid for every  $\sigma \in S_0$  and with  $S_\sigma$  denoting the class of all  $\{\tilde{F}_t\}$ -stopping times that satisfy  $\tilde{P}[\sigma \leq \tau < \infty] = 1$ . The stopping time

$$(5.9) \quad \sigma_t = \begin{cases} \inf\{s \geq t; \xi_s = Q_s\} \\ \infty, \text{ if } \{\dots\} \end{cases} \in S_t^*$$

is optimal for the problem (5.6):  $u(t) = \tilde{E}(Q_{\sigma_t} | \tilde{F}_t)$ , and we have

$$(5.10) \quad \xi_\infty = Q_\infty, \quad \text{a.s. } \tilde{P}.$$

On the other hand, the continuous, nondecreasing process  $\Lambda = \{\Lambda_t, \tilde{F}_t; 0 \leq t < \infty\}$  in the Doob-Meyer decomposition (4.16) is integrable, and the right-continuous martingale  $M = \{M_t, \tilde{F}_t; 0 \leq t < \infty\}$  is uniformly integrable and admits the (Fujisaki-Kallianpur-Kunita) representation

$$(5.11) \quad M_t = \tilde{E}(M_0) + \sum_{j=1}^d \int_0^t \phi_j(s) dB_s^{(j)}; \quad 0 \leq t < \infty$$

as an almost surely continuous process. Here  $\{\phi_j(t), \tilde{F}_t; 0 \leq t < \infty\}$  are measurable, adapted processes with

$$(5.12) \quad \tilde{P}\left[\int_0^T \phi_j^2(t) dt < \infty\right] = 1; \quad 0 \leq T < \infty, \quad 1 \leq j \leq d;$$

see, for instance, [18], Theorem 5.20. Finally, the process

$$(5.13) \quad v_t \triangleq \frac{1}{\xi_t} \left( \xi_t - \int_0^t \xi_s g_s ds \right), \tilde{F}_t; \quad 0 \leq t < \infty$$

is easily seen to satisfy the tenets of Definition 5.2; it is the value of the portfolio  $\Pi$  determined by (4.27), where now

$$(5.14) \quad \eta(t) \triangleq \sigma(t) H^{-1}(t) \phi(t), \tilde{F}_t; \quad 0 \leq t < \infty.$$

It is easily seen (using (5.12)) that this portfolio satisfies the requirement (5.2) and that the optimal stopping time  $\tau_t$  for (5.6) coincides with  $\hat{\tau}_t$  of (3.8).

5.4 The Markovian case: If the processes in (2.6) are equal to the constants  $\sigma_{ij}, r, a_i, \mu_i$  respectively, then the stock price process

$$\tilde{X} = (X^{(1)}, \dots, X^{(n)})^*$$

satisfies the system (2.3) of linear equations with constant coefficients. If furthermore, the processes  $F$  and  $G$  of Definition 3.1 are given as

$$f_t = \phi(\tilde{X}_t), g_t \equiv 0; \quad 0 \leq t < \infty$$

for some continuous function  $\phi: \mathbb{R}_+^n \rightarrow [0, \infty)$ , then under the conditions of Theorem 4.2 the valuation process is obtained via

$$(5.15) \quad V_t = v(\tilde{X}_t); \quad 0 \leq t < \infty, \quad \text{a.s. } \tilde{P}$$

where  $v: \mathbb{R}_+^n \rightarrow [0, \infty)$  is the least  $r$ -excessive majorant of  $\phi$  (see [6]).

5.5 Remark: In the case of Example 3.3 with  $\sigma_{11}(t) \equiv \sigma > 0$ ,  $\mu(t) \geq \mu > 0$ , we have

$$Q_t = E_t(X_t^{(1)} - c)^+ \leq E_t X_t^{(1)}$$

where the last process satisfies (2.17). It follows easily from this equation that

$$0 \leq Q_t \leq x e^{\sigma Y_t}; \quad 0 \leq t < \infty$$

holds a.s.  $\tilde{P}$ , where  $Y_t = B_t^{(1)} - \nu t$  is Brownian motion with negative drift and  $\nu = \frac{\mu}{\sigma} + \frac{\sigma}{2}$ . But now the law

$$\tilde{P}[\sup_{0 \leq t < \infty} Y_t \in db] = 2\nu e^{-2\nu b} db; \quad b > 0$$

is well-known, and the condition (5.1) follows from it; consequently, Theorem 5.3 applies to such perpetual American options.

For an American option as above but with  $\mu(t) \equiv 0$  (i.e., on a stock which pays no dividends) and with

$$(5.16) \quad r_t \geq 0, \lim_{T \rightarrow \infty} \int_0^T r_s ds = \infty; \quad \text{a.s. } \tilde{P},$$

we shall agree that the valuation process is given by

$$(5.17) \quad V_t \stackrel{\Delta}{=} \lim_{\mu \downarrow 0} V_t(\mu); \quad 0 \leq t < \infty$$

where  $V(\mu)$  is the valuation process under a constant dividend rate, provided that the limit in (5.17) exists almost surely. From the optional sampling theorem we obtain

$$\begin{aligned} \beta_t X_t^{(1)} &\geq \operatorname{esssup}_{\tau \in S_t^*} \tilde{E}[X_\tau^{(1)} \beta_\tau | \tilde{F}_t] \geq V_t(\mu) \beta_t \\ &= \operatorname{esssup}_{\tau \in S_t^*} \tilde{E}[(X_\tau^{(1)} - c)^+ \beta_\tau | \tilde{F}_t] \\ &\geq \tilde{E}[(X_T^{(1)} - c)^+ \beta_T | \tilde{F}_t] \\ &\geq \beta_t X_t^{(1)} e^{-\int_t^T r_s ds} - c \tilde{E}[\beta_T | \tilde{F}_t] \end{aligned}$$

for every  $\mu > 0$ ,  $T > t$ . Letting  $\mu \downarrow 0$  and then  $T \rightarrow \infty$ , we obtain  $V_t = X_t^{(1)}$  from (5.17). In words, a perpetual American option on a stock which pays no dividends, and in the presence of condition (5.16), must sell for the same amount as the stock.

If the perpetual option has zero exercise price, the same conclusion holds without the restriction  $\mu(t) \equiv 0$  (by analogy with Remark 4.7).

5.6 Example: In the Markovian case 5.4 with  $d = n = 1$ ,  $c = 1$ ,  $\phi(x) = (x - 1)^+$ , the function  $v$  in (5.15) was computed by McKean [11] as

$$v(x) = \begin{cases} (\kappa - 1) \left(\frac{x}{\kappa}\right)^\gamma; & 0 < x < \kappa \\ x - 1; & x \geq \kappa \end{cases}$$

with  $\gamma = \frac{1}{\sigma^2} (\sqrt{\delta^2 + 2r\sigma^2} - \delta)$ ,  $\alpha = r - \mu > 0$ ,  $\delta = \alpha - \frac{\sigma^2}{2}$ ,  $\kappa = \frac{\gamma}{\gamma - 1} > 1$ , and the optimal exercise time (5.9) becomes  $\rho_t = \inf\{s \geq t; X_s \geq \kappa\} \in S_t$ . The finite-horizon version of this problem was studied by Van Moerbeke [17], along with the associated free boundary problem.

## 6. References

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